

PRIME NUMBERS WITH A CERTAIN EXTREMAL TYPE PROPERTY

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ABSTRACT. The convex hull of the subgraph of the prime counting function $x \rightarrow \pi(x)$ is a convex set, bounded from above by a graph of some piecewise affine function $x \rightarrow \epsilon(x)$. The vertices of this function form an infinite sequence of points $(e_k, \pi(e_k))_1^\infty$. In this paper we present some trivial observation about the sequence $(e_k)_1^\infty$ and we formulate a number of questions resulting from the numerical data. Besides we prove one less trivial result: *if the Riemann hypothesis is true, then $\lim_{e_k} \frac{e_{k+1}}{e_k} = 1$.*

1. INTRODUCTION

Prime numbers are generators of the multiplicative semigroup \mathbb{N}^* (where $\mathbb{N}^* = \{1, 2, 3, \dots\}$). It is well known, that it is impossible to distinguish two different prime numbers using only the "language of multiplication". If one wants to distinguish some particular prime number from the others, one must consider an additional structure in \mathbb{N}^* , like for example the natural order in \mathbb{N} . The prime counting function is an example of such order properties. In this paper we define a property of prime numbers with respect to their position on the graph of the prime counting function $x \rightarrow \pi(x)$.

Some properties related to the graph of the function π were studied several years ago in 1979 by Carl Pomerance [1] and recently (2006) by H.L. Montgomery and S.Wagon [2] in considerations concerning the Prime Number Theorem (PNT for short).

Let \mathbb{P} denote the sequence of prime numbers, i.e. $\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$. Usually one defines the function $\pi : [2, \infty) \rightarrow [1, \infty)$ by the formula

$$(1) \quad \pi(x) = \sum_{p \in \mathbb{P}, p \leq x} 1.$$

For our purposes it will be a little more convenient to consider a function $\pi^* : [2, \infty) \rightarrow [1, \infty)$ defined as follows. First we define a continuous function $\eta : [1, \infty) \rightarrow [2, \infty)$ setting: $\eta(n) = p_n$, where p_n is the n -th prime number, and η is affine (and continuous) in the intervals $[n, n+1]$ for each $n \in \mathbb{N}$. Obviously η is strictly increasing, continuous and surjective. Thus η is invertible and we define π^* as the inverse of η . Let $[x]$ denote the integral part of the real number x . One can easily check, that π and π^* have the same values at prime numbers, and that

$$(2) \quad \pi(x) = [(\pi^*(x))].$$

2. PART I

2.1. Definition of extremal primes. The function π^* is increasing, continuous, but it is "visibly" not concave. However there are many concave functions $\varphi : [2, \infty) \rightarrow [1, \infty)$, such

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that for each $x \in [2, \infty)$ we have $\varphi(x) \geq \pi^*(x)$. This follows for example from the Chebyshev theorem, which gives the inequality

$$(3) \quad A \cdot \frac{x}{\ln(x)} < \pi(x) < B \cdot \frac{x}{\ln(x)}$$

for some $A < 1$ and $B > 1$, (obviously $\frac{x}{\ln x}$ is a concave function).

Let us consider the set

$$(4) \quad \Omega = \{f : [2, \infty) \rightarrow [1, \infty) : f \geq \pi^*, f - \text{concave}\},$$

and let us observe, although this will play no role in our consideration, that Ω is a subset of the vector cone of all positive and concave real functions on $[2, \infty)$.

We put for $x \in [2, \infty)$

$$(5) \quad \epsilon(x) = \inf \{f(x) : f \in \Omega\},$$

i.e. the function ϵ is the lower envelope of the family Ω . In other words the function ϵ is the smallest concave function, which is greater than π^* (equivalently than π). Since π^* is piecewise affine, then ϵ is also the lower envelope of those functions from Ω , which are piecewise affine. Then it is clear, that the function ϵ is concave and it is also piecewise affine. Thus the set

$$(6) \quad \Gamma = \{(x, y) \in \mathbb{R}^2 : x \in [2, \infty), 0 \leq y \leq \epsilon(x)\}$$

is a convex set. Let us recall, that if U is a convex set and $b \in U$, then b is said to be an *extremal point* of U iff b is not an interior point of any non-trivial segment lying in U .

Now we are ready to formulate the following:

Definition 1. *The prime number $p \in \mathbb{P}$ will be said to be extremal prime number, when the point $(p, \pi(p))$ is an extremal point of the convex set Γ .*

2.2. Properties of the set of extremal primes. Let \mathbb{E} denote the set of all extremal primes. Sometimes we will think rather about the sequence of extremal primes $\mathbb{E} = \{e_1, e_2, \dots\}$, where $e_1 < e_2 < e_3, \dots$, i.e. the sequence $(e_k)_1^\infty$ is strictly increasing.

Now we will present some easy properties of the set \mathbb{E} .

Proposition 2. *The set \mathbb{E} is not empty.*

Indeed, it is easy to check, that $2 \in \mathbb{E}$.

Proposition 3. *The set $\mathbb{N}^* \setminus \mathbb{E}$ is not empty.*

One can check, that $3 \in \mathbb{E}$, $7 \in \mathbb{E}$, but $5 \notin \mathbb{E}$.

Proposition 4. *The set \mathbb{E} is infinite.*

Proof. Let l_k denote the straight line (the affine function) passing through the points $(e_{k-1}, \pi(e_{k-1}))$ and $(e_k, \pi(e_k))$. It follows from Definition 1 that the graph of the function ϵ lies below the line l_k . This gives a simple inductive method of finding the next extremal prime e_{k+1} providing, that we know $e_1, e_2, \dots, e_{k-1}, e_k$ (in fact it is sufficient to know only e_{k-1} and e_k). We can do it as follows. We consider the difference quotients of the form

$$(7) \quad I_k(p) = \frac{\pi(p) - \pi(e_k)}{p - e_k},$$

for $p \in \mathbb{P}, p > e_k$. It follows from the remark made above, that for each $p > e_k$ we have:

$$(8) \quad 0 < I_k(p) < \frac{\pi(e_k) - \pi(e_{k-1})}{e_k - e_{k-1}} = I_{k-1}(e_k).$$

Using the commonly known fact

$$(9) \quad \lim_{p \rightarrow \infty} \frac{\pi(p)}{p} = 0$$

we have $\lim_{p \rightarrow \infty} I_k(p) = 0$. Then there exists a finite set $\mathbb{P}_k \subset \mathbb{P}$ of primes, such that $p_o \in \mathbb{P}_k \implies p_o > e_k$ and such that $I_k(p) \leq I_k(p_o)$ for $p > e_k$. We set then $e_{k+1} = \max \mathbb{P}_k$. This implies, that the set \mathbb{E} is infinite. □

Proposition 5. *The derivative $x \longrightarrow \epsilon'(x)$ is strictly decreasing and tends to 0 at infinity.*

Proof. Let

$$(10) \quad \delta_k = \frac{\pi(e_{k+1}) - \pi(e_k)}{e_{k+1} - e_k},$$

i.e. δ_n is the slope of the n -th segment lying on the graph of the function ϵ . Since ϵ is increasing and concave, then the sequence $(\delta_k)_1^\infty$ is positive and strictly decreasing. Let us observe, that the sequence $(\delta_k)_1^\infty$ may be identified with the derivative of the function ϵ . Hence the limit $\delta = \lim_{k \rightarrow \infty} \delta_k \geq 0$ exists and it must be $\delta = 0$, which follows once more from (9). □

The number $\alpha_k = \delta_k^{-1}$ is a measure of the density of prime numbers in the interval $[e_k, e_{k+1})$ and may be interpreted as an *average gap* between primes in $[e_k, e_{k+1})$. By the remark made above, the sequence $(\alpha_k)_1^\infty$ is strictly increasing.

It is natural to ask now about the cardinality of the set $\mathbb{N} \setminus \mathbb{E}$. We have

Proposition 6. *The set $\mathbb{N} \setminus \mathbb{E}$ is infinite.*

Proof. This is true and is related to study of *small gaps between primes*. Let us observe only, that the finiteness of $\mathbb{N} \setminus \mathbb{E}$ is impossible if the *twin primes* conjecture is true. However, we know now from the recent result of Zhang, [3] that $\liminf(p_{n+1} - p_n) < 7 \cdot 10^7$. It follows from Proposition 5 that this is sufficient for the set $\mathbb{N} \setminus \mathbb{E}$ to be infinite. □

It appears, that the set \mathbb{E} is in some sense minimal with respect to Property 5. Namely suppose, that $\mathbb{G} = (g_i)_1^\infty$ is a subsequence of the sequence \mathbb{P} of prime numbers such that $g_1 = 2$. Let

$$(11) \quad \delta_k(\mathbb{G}) = \frac{\pi(g_{k+1}) - \pi(g_k)}{g_{k+1} - g_k}.$$

We will say, that \mathbb{G} is concave, when $\delta_k(\mathbb{G})$ is strictly decreasing. For example the sequence \mathbb{E} is concave, while the sequence \mathbb{P} is not concave. A subsequence of a concave sequence is also concave. The sequence \mathbb{E} of extremal primes has the following property: *if \mathbb{E} is a subsequence of a concave sequence \mathbb{G} , then $\mathbb{E} = \mathbb{G}$.* More exactly:

Proposition 7. *Let us suppose that a sequence $(g_k)_1^\infty$ is concave and the sequence \mathbb{E} is a subsequence of \mathbb{G} . Then $\mathbb{E} = \mathbb{G}$.*

Proof. Clearly $e_1 = g_1 = 2$. Since there are no primes between 2 and 3 and $e_2 \in \mathbb{G}$ then also $e_2 = g_2 = 3$. Suppose now that $e_i = g_i$ for $1 \leq i \leq k$. We wish to prove, that $e_{k+1} = g_{k+1}$. Assume then, that $e_{k+1} \neq g_{k+1}$ and that $g_{k+m} = e_{k+1}$ i.e. that

$$e_k = g_k < g_{k+1} < g_{k+2} < \dots < g_{k+m} = e_{k+1}.$$

Now, using the notations from Proposition 4 and the definition of e_{k+1} we have for $i < m$:

$$(12) \quad \delta_k(\mathbb{G}) = I_k(g_{k+1}) < \delta_k(\mathbb{E})$$

Let us consider a function $H : [e_k, e_{k+1}] \rightarrow \mathbb{R}$ such that $H(g_{k+i}) = \pi(g_{k+i})$ and H is affine and continuous in each interval $[g_{k+i}, g_{k+i+1}]$. We see, that the function H is continuous and differentiable except in the points $x = g_{k+i}$ and its derivative in the intervals (g_{k+i}, g_{k+i+1}) is constant and equal $\delta_{k+i}(\mathbb{G})$. It follows from our assumptions (since \mathbb{G} is concave), that

$$(13) \quad \sup \left\{ H'(x) : x \in [e_k, e_{k+1}] \right\} = \delta_k(\mathbb{G}) < \delta_k(\mathbb{E}).$$

Let us observe, that since the function H is continuous and differentiable except for a finite set of arguments, we can apply the mean value theorem. Hence we have:

$$\begin{aligned} \pi(e_{k+1}) - \pi(e_k) = \pi(g_{k+p}) - \pi(g_k) &\leq \sup \left\{ (H'(x) : x \in [e_k, e_{k+1}]) \right\} \cdot (g_{k+p} - g_k) \\ &\leq \delta_k(\mathbb{G}) \cdot (e_{k+1} - e_k) < \delta_k(\mathbb{E}) \cdot (e_{k+1} - e_k) = \pi(e_{k+1}) - \pi(e_k), \end{aligned}$$

but this is impossible and this ends the proof of Proposition 7. □

2.3. Some numerical data and the questions they evoke. The observations about the extremal primes made above are rather trivial. We will prove later some deeper, however conditional, results. We have calculated the first 2200 extremal primes and after studying these numerical data, we can formulate a number of more or less interesting questions. It is impossible to give here the complete list of the first 2200 extremal primes, but we will present some selected data:

The first twenty eight terms of the sequence \mathbb{E} are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
e_n	2	3	7	19	47	73	113	199	283	467	661	887	1129	1329

n	15	16	17	18	19	20	21	22	23	24	25	26	27	28
e_n	1627	2803	3947	4297	5881	6379	7043	9949	10343	13187	15823	18461	24137	33647

The list of e_k where $k \leq 2200$ and $k \equiv 0 \pmod{100}$:

e_{100}	5253173
e_{200}	67596937
e_{300}	314451367
e_{400}	883127303
e_{500}	2122481761
e_{600}	4205505103
e_{700}	7274424463
e_{800}	12251434927
e_{900}	19505255383
e_{1000}	28636137347
e_{1100}	40001601779
e_{1200}	55036621907
e_{1300}	73753659461
e_{1400}	97381385771
e_{1500}	125232859691
e_{1600}	157169830847
e_{1700}	196062395777
e_{1800}	241861008029
e_{1900}	296478801431
e_{2000}	365234091199
e_{2100}	435006680401
e_{2200}	524320812671

The examination of the sequence of the first 2200 extremal primes allows us to formulate a number of questions. First of all it seems to be interesting to say something about the "density" of the sequence \mathbb{E} . Our "experimental" data support some conjectures. Namely

Conjecture 8. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{e_k}$$

is convergent.

It follows from our data that

$$\sum_{k=1}^{2000} \frac{1}{e_k} \cong 1,090..$$

Conjecture 9. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{\ln e_k}$$

is divergent.

Our data gives:

$$\sum_{k=1}^{2000} \frac{1}{\ln e_k} > 100.$$

Since the set \mathbb{E} of extremal prime numbers is infinite and, clearly, the problem of finding any reasonable explicit formula describing the correspondence $\mathbb{N} \ni n \longrightarrow e_n$ is rather hopeless, we may define and try to study a function, which may be called *extremal primes counting function* π_{ϵ} . The formula for π_{ϵ} is analogous to the Formula (1). We set

$$(14) \quad \pi_\epsilon(x) = \sum_{p \in \mathbb{E}, p \leq x} 1.$$

Unfortunately we know only 2200 values of $\pi_\epsilon(x)$ for $x \leq 5 \cdot 10^{11}$. However it seems to be possible to formulate some conjectures about π_ϵ . Clearly $\pi_\epsilon(x) \leq \pi(x)$ and the growth of π_ϵ is much slower than the growth of π . For example $\pi_\epsilon(x_o) = 1700$, when $x_o = 196062395777$ and for the same x_o we have $\pi(x_o) = 7855721212$. In particular we may try to find the best $\alpha < 1$ such that $\pi_\epsilon(x) = o(x^\alpha)$ observing the ratio $\frac{\ln n}{\ln e_n}$ when n tends to infinity (in our case only to $n \leq 5 \cdot 10^{11}$). May be only accidentally, but the best α obtained from our data is near to $\frac{\gamma}{2}$, where γ is the Euler constant. Hence we formulate:

Conjecture 10. *There exists infimum*

$$\inf \{ \alpha > 0 : \pi_\epsilon(x) = o(x^\alpha) \}$$

and it is positive.

Our numerical data support strongly also the following interesting conjecture:

Conjecture 11. *In the notations as above, we have:*

$$\lim_{k \rightarrow \infty} \frac{e_{k+1}}{e_k} = 1.$$

We will prove below, in Part II, that the Riemann Hypothesis implies the Conjecture 11. This conjecture is interesting itself, but also because of the following:

Proposition 12. *If*

$$\lim_{k \rightarrow \infty} \frac{e_{k+1}}{e_k} = 1$$

then

$$\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 1.$$

Proof. For each $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that

$$e_{k(n)} \leq p_n < p_{n+1} \leq e_{k(n)+1}.$$

Thus

$$\frac{p_{n+1}}{p_n} \leq \frac{e_{k(n)+1}}{e_{k(n)}}$$

and the last sequence tends by our assumption to 1. Let us recall here, that $\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 1$ implies PNT.

□

It follows directly from the definitions of the functions π and π_ϵ that $\pi(e_{k+1}) - \pi(e_k) \geq 1$ and the equality may occur. Except for trivial $e_1 = 2$ and $e_2 = 3$ I have found two such "twin extremal primes" for $k = 116$ and $k = 976$. Namely: $e_{116} = 8787901$, $e_{117} = 8787917$ and $\pi(e_{116}) = 589274$, $e_{976} = 26554262369$, $e_{977} = 26554262393$ and $\pi(e_{976}) = 1156822345$. We ask if:

Question 13. *Does there exists infinitely many $k \in \mathbb{N}$ such that $\pi(e_{k+1}) - \pi(e_k) = 1$.*

Some additional remarks about the "small" gaps between extremal primes are in Part III.

Another exception is related to the inequality $I_k(p) \leq I_k(p_o)$, which is described in Proposition 4. One may ask if the number of points $p > e_k$ such that $I_k(p) = I_k(p_o)$ is greater than 1. In our numerical data we have only two such examples, namely for $k = 2$ we have $I_2(5) = I_2(7)$ and also $I_4(23) = I_4(31) = I_4(43) = I_4(47) = \frac{1}{4} = \delta_4$ but in fact our programme searching "next extremal primes" was not written to "catch" such exceptions.

3. PART II

3.1. Definition of lenses. With the notation as in Part I, the intervals $[e_k, e_{k+1})$ (in \mathbb{N}) will be called *lenses*. More exactly:

Definition 14. Definition: Given a positive integer $k \in \mathbb{N}$ the lens S_k is a set

$$S_k = \{n \in \mathbb{N} : e_k \leq n < e_{k+1}\}.$$

The difference $e_{k+1} - e_k$ will be called the length of the lens S_k and will be denoted by $|S_k|$.

Sometimes we will use the name "lens" for a part of graph of π^* for $x \in [e_k, e_{k+1})$. Our aim is to study the order of magnitude of $|S_k|$ when $k \rightarrow \infty$. Since we will apply the language of differential calculus, it will be more comfortable to work with the function $[2, \infty) \ni x \rightarrow S(x) \in [1, \infty)$ where

$$x \in [e_k, e_{k+1}) \implies S(x) = |S_k|.$$

The typical lenses and the graph of $\epsilon(x)$ for $x \leq 113$ are illustrated on the pictures 1-3 at the end of this paper.

3.2. The integral logarithm and error term. We shall consider the following - well known - functions: $L : [2, \infty) \rightarrow [0, \infty)$ and $\epsilon : [2, \infty) \rightarrow [0, \infty)$, defined by the following formulas:

$$(15) \quad L(x) = \int_2^x \frac{1}{\ln t} dt$$

and

$$(16) \quad \epsilon(x) = \sqrt{x} \cdot \ln x.$$

The first is called *integral logarithm* (we will write also $L(x) = Li(x)$), and the second is called *error term*. Together with L and ϵ we will consider the functions

$$(17) \quad \varphi(x) = L(x) - \epsilon(x)$$

and for $x \in (2, \infty)$ and $h \in \mathbb{R}$

$$(18) \quad l(x, h) = \varphi'(x) \cdot h + \varphi(x)$$

Clearly all these functions are analytic at least in $(2, \infty)$. We will use the derivatives of the considered functions to the order four and we shall write y instead of $\ln x$ to present some formulas in more compact form. Hence we have:

$$(19) \quad L^{(1)}(x) = \frac{1}{\ln x} = \frac{1}{y}$$

$$(20) \quad L^{(2)}(x) = \frac{-1}{x \cdot \ln x} = \frac{-1}{x \cdot y^2}$$

$$(21) \quad L^{(3)}(x) = \frac{\ln x + 2}{x^2 \cdot \ln^3 x} = \frac{y + 2}{x^2 \cdot y^3},$$

$$(22) \quad L^{(4)}(x) = \frac{-(2 \cdot \ln^2 x + 6 \ln x + 6)}{x^3 \cdot \ln^4 x} = \frac{-(2 \cdot y^2 + 6y + 6)}{x^3 \cdot y^4}$$

The derivatives of error term function, written in an analogous manner, run as follows:

$$(23) \quad \epsilon(x) = \sqrt{x} \cdot \ln x = \sqrt{x} \cdot y,$$

$$(24) \quad \epsilon^{(1)}(x) = \frac{\ln x + 2}{2\sqrt{x}} = \frac{y + 2}{2\sqrt{x}},$$

$$(25) \quad \varepsilon^{(2)}(x) = \frac{-\ln x}{4x\sqrt{x}} = \frac{-y}{4x\sqrt{x}}$$

$$(26) \quad \varepsilon^{(3)} = \frac{3\ln x - 2}{8x^2\sqrt{x}} = \frac{3y - 2}{8x^2\sqrt{x}},$$

$$(27) \quad \varepsilon^{(4)}(x) = \frac{-15\ln x + 16}{16x^3\sqrt{x}} = \frac{-15y + 16}{16x^3\sqrt{x}}.$$

Let us observe, that the second derivatives of the functions L and ε are negative, so both these functions are concave.

The second derivative of the function φ has the form

$$\varphi^{(2)}(x) = \frac{-4\sqrt{x} + \ln^3 x}{x\sqrt{x}\ln^2 x} = \frac{-4\sqrt{x} + y^3}{4x\sqrt{xy^2}}$$

then taking into account that

$$\lim_{x \rightarrow \infty} (-4\sqrt{x} + \ln^3 x) = -\infty$$

we can state :

Proposition 15. *There exists $x_o \in (2, \infty)$ such, that the function φ is concave in the interval $[x_o, \infty)$.*

3.3. A remark on Taylor polynomials of considered functions. Let us fix a point $x \in (2, \infty)$. Let $T_{x,L}^{(3)}$ denote the Taylor polynomial of order three of the function L with the center at x . Hence

$$(28) \quad T_{x,L}^{(3)}(h) = L(x) + L^{(1)}(x) \cdot h + \frac{1}{2} \cdot L^{(2)}(x) \cdot h^2 + \frac{1}{6} \cdot L^{(3)}(x) \cdot h^3.$$

The remainder $R_x^{(3)}(h) = L(x+h) - T_{x,L}^{(3)}(h)$, written in the Lagrange form, is given by the formula:

$$(29) \quad R_x^{(3)}(h) = \frac{1}{24} L^{(4)}(\xi) \cdot h^4,$$

where ξ is a point from the $(x, x+h)$. Since $L^{(4)} < 0$ in all its domain, we have the inequality:

Proposition 16. *For each $x \in (2, \infty)$ and for each $h \in (2-x, \infty)$ the following inequality is true:*

$$L(x+h) \leq T_{x,L}^{(3)}(h).$$

Let $T_{x,\varepsilon}^{(3)}$ denote the Taylor polynomial of order three of the function ε with the center at x , i.e.

$$(30) \quad T_{x,\varphi}^{(3)}(h) = \varepsilon(x) + \varepsilon^{(1)}(x) \cdot h + \frac{1}{2} \cdot \varphi^{(2)}(x) \cdot h^2 + \frac{1}{6} \cdot L^{(3)}(x) \cdot h^3.$$

Using an analogous argumentation as in the case of the function L we have:

Proposition 17. *For each $x \in (2, \infty)$ and for each $h \in (2-x, \infty)$ the following inequality is true:*

$$\varepsilon(x+h) \leq T_{x,\varepsilon}^{(3)}(h),$$

and in consequence we have the inequality (true for all $h \in (2 - x, \infty)$):

$$(31) \quad L(x + h) + \varepsilon(x + h) < T_{x,L}^{(3)}(h) + T_{x,\varepsilon}^{(3)}(h)$$

3.4. Definition of two functions. In this section we shall define two functions $h_+ : (x_o, \infty) \ni x \rightarrow h_+(x) \in \mathbb{R}$ and $h_- : (x_o, \infty) \ni x \rightarrow h_-(x) \in \mathbb{R}$, where x_o is the point defined in Proposition 15. First we will describe in details the definition of the function h_+ . The definition of h_- will be similar.

Let us fix a point $x \in (x_o, \infty)$. Take into account the tangent line $l(x, h)$ to the graph of the function φ at the point $(x, \varphi(x))$. Its equation for $h \in \mathbb{R}$ is given by:

$$(32) \quad l(x, h) = \varphi'(x) \cdot h + \varphi(x) = L'(x)h - \varepsilon'(x)h + L(x) - \varepsilon(x).$$

The "tangent half-lines" obtained, when we restrict ourselves in the Formula (32) to $h \in [0, \infty)$ or $h \in (-\infty, 0]$ will be denoted by $l_+(x, h)$ or $l_-(x, h)$ respectively.

For $h = 0$ we have the inequality:

$$l(x, 0) = \varphi(x) = L(x) - \varepsilon(x) < L(x) + \varepsilon(x).$$

This means that the half-line l_+ "starts" from the interior point $(x, \varepsilon(x))$ of the subgraph of the function $L + \varphi$, which is a convex set. Since

$$\frac{d}{dh} L(x + h) = \frac{1}{\ln(x + h)}$$

and

$$\frac{d}{dh} \varepsilon(x + h) = \frac{\ln(x + h) + 2}{2\sqrt{x + h}}$$

then

$$\lim_{h \rightarrow \infty} \frac{d}{dh} (L(x + h) + \varepsilon(x + h)) = 0.$$

On the other hand

$$\frac{d}{dh} l(x + h) = \varphi'(x) > 0,$$

hence the half-line $l_+(x, h)$ must intersect the graph of the strictly concave function $L(x + h) + \varepsilon(x + h)$ in exactly one point. Hence we have proved the following:

Proposition 18. *For each $x \in (x_o, \infty)$ there exists exactly one positive number $h_+(x)$ such that*

$$L(x + h_+(x)) + \varepsilon(x + h_+(x)) = \varphi'(x) \cdot h_+(x) + \varphi(x).$$

In other words for each $x \in (x_o, \infty)$ the equation (with unknown h):

$$(33) \quad L(x + h) + \varepsilon(x + h) = \varphi'(x) \cdot h + \varphi(x)$$

has exactly one positive solution, which we will denote by $h_+(x)$.

If one replaces the half-line $l_+(x, h)$, by the half line $l_-(x, h)$, then applying the same arguments as above, we obtain:

Proposition 19. *For each $x \in (x_o, \infty)$ there exists exactly one negative number $h_-(x)$ such that*

$$L(x + h_-(x)) + \varepsilon(x + h_-(x)) = \varphi'(x) \cdot h_-(x) + \varphi(x).$$

In other words equation (33) has exactly one negative solution, which we will denote by $h_-(x)$.

3.5. An auxiliary equation. .

In this paper we would like to establish the order of magnitude of the functions $x \rightarrow h_+(x)$ and $x \rightarrow h_-(x)$ (in fact of the difference $h_+(x) - h_-(x)$), when x tends to $+\infty$. Since the equation (33) is rather hard to solve, we will consider an auxiliary equation:

$$(34) \quad T_{x,L}^{(3)}(h) + T_{x,\varepsilon}^{(3)}(h) = \varphi'(x) \cdot h + \varphi(x),$$

which can be written in the form:

$$(35) \quad W_x(h) := \frac{1}{6}(L^{(3)}(x) + \varepsilon^{(3)}(x)) \cdot h^3 + \frac{1}{2}(L^{(2)}(x) + \varepsilon^{(2)}(x)) \cdot h^2 + 2\varepsilon^{(1)}(x) \cdot h + 2\varepsilon(x) = 0.$$

As we see, equation (35) is an algebraic equation of degree three. It has at least one real root. We will see that it can have (and has) more then one real root. We will be interested not only on the existence of roots of equation (35), but also on theirs signs. Let us observe, that since $W_x(0) = 2\varepsilon(x) > 0$ then the number $h = 0$ cannot be a root of considered equation. Let us also observe that, in fact, equation (35) is not a single algebraic equation, but it is a one parameter family of algebraic equations, where the parameter is $x \in (x_o, \infty)$.

We will prove the following :

Lemma 20. *i). There exists $x_+ \in (x_o, \infty)$, such that for each $x > x_+$ the equation $W_x(h) = 0$ has a positive root.*

ii). There exists $x_- \in (x_o, \infty)$, such that for each $x > x_-$ the equation $W_x(h) = 0$ has a negative root.

The proof of the lemma is done together with the proof of Proposition 25. Assume now, that Lemma 20 is true. This allows us to define two new functions h_+^* and h_-^* . We will describe in details the definition of h_+^* . We set

Definition 21. *Let $x \in (x_+, \infty)$. Then the set of positive roots of equation (??) is not empty and we set:*

$$h_+^*(x) = \min \{h > 0 : W_x(h) = 0\}.$$

The relation between the functions h_+ and h_+^* is the following:

Proposition 22. *If Lemma 20 is true, then for $x \in (x_+, \infty)$ we have the inequality: $h_+(x) < h_+^*(x)$.*

Proof. Let us fix $x \in (x_+, \infty)$. In the interval $[x, x + h_+(x)]$, i.e. for $h \in [0, h_+(x)]$ the line $l(x, h)$ lies below the graph of the function $L + \varepsilon$. This follows directly from the definition of the function $h_+(x)$. Hence in this interval the line $l(x, h)$ cannot intersect the graph of the function $T_{x,\varepsilon}^{(3)} + T_{x,L}^{(3)}$ because of inequality (31). Hence the equation $W_x(h) = 0$ has no roots in the interval $h \in [0, h_+(x)]$. But this means that $h_+(x) < h_+^*(x)$, which ends the proof of Proposition 22. \square

Assume once more, that Lemma 20 is true. We have

Definition 23. *Let $x \in (x_-, \infty)$. Then the set of negative roots of equation (35) is not empty and we set:*

$$h_-^*(x) = \max \{h < 0 : W_x(h) = 0\}.$$

The relation between the functions h_- and h_-^* is as follows:

Proposition 24. *If Lemma (20) is true, then for $x \in (x_-, \infty)$ we have the inequality: $h_-(x) > h_-^*(x)$.*

The proof of Proposition 24 is similar to the proof of Proposition 22.

3.6. The proof of the main lemma. Now we will prove Lemma (20). Equation (35) we are interested in, can be written in the form:

$$(36) \quad A_3(x) \cdot h^3 + A_2(x) \cdot h^2 + A_1(x) \cdot h + A_o(x) = 0$$

where, using formulas 21-28, we have:

$$(37) \quad A_3(x) = \frac{1}{6}(L^{(3)}(x) + \varepsilon^{(3)}(x)) = \frac{1}{48} \cdot \frac{8\sqrt{x}(y+2) + y^3(3y-2)}{x^2\sqrt{xy}^3},$$

$$(38) \quad A_2(x) = \frac{1}{2}(L^{(2)}(x) + \varepsilon^{(2)}(x)) = \frac{-1}{8} \cdot \frac{4\sqrt{x} + y^3}{x\sqrt{xy}^2}.$$

$$(39) \quad A_1(x) = \frac{y+2}{\sqrt{x}},$$

$$(40) \quad A_o(x) = 2\sqrt{xy}.$$

Now, taking into account the fact, that for x sufficiently large $A_3(x) > 0$, we divide equation (36) by $A_3(x)$ in order to obtain the form:

$$(41) \quad h^3 + B_2(x) \cdot h^2 + B_1(x) \cdot h + B_o(x) = 0$$

where

$$(42) \quad B_2(x) = \frac{A_2(x)}{A_3(x)} = -6x \frac{4\sqrt{xy} + y^4}{8\sqrt{xy} + 16\sqrt{x} + 3y^4 - 2y^3},$$

$$(43) \quad B_1(x) = \frac{A_1(x)}{A_3(x)} = 48x^2 \frac{y^3}{8\sqrt{xy} + 16\sqrt{x} + 3y^4 - 2y^3},$$

$$(44) \quad B_o(x) = \frac{A_o(x)}{A_3(x)} = 96x^3 \frac{y^4}{8\sqrt{xy} + 16\sqrt{x} + 3y^4 - 2y^3}.$$

For further analysis of equation 41 it will be convenient to use some Landau symbols. Let us recall that for a function g defined in the neighbourhood of $+\infty$ one writes $g = o(1)$ if and only if $\lim_{x \rightarrow +\infty} g(x) = 0$. Using this convention, we can write:

$$(45) \quad B_2(x) = -6x \frac{\frac{1}{2} + o(1)}{1 + o(1)},$$

$$(46) \quad B_1(x) = 48x^2 \frac{o(1)}{1 + o(1)},$$

$$(47) \quad B_o(x) = 96x^3 \frac{o(1)}{1 + o(1)}.$$

This makes possible to write equation 41 in the form:

$$(48) \quad h^3 - 6x \frac{\frac{1}{2} + o(1)}{1 + o(1)} h^2 + 48x^2 \frac{o(1)}{1 + o(1)} h + 96x^3 \frac{o(1)}{1 + o(1)} = 0.$$

Now we apply the substitution $h = \theta x$, which leads to the form:

$$(49) \quad \theta^3 x^3 - 6x \frac{\frac{1}{2} + o(1)}{1 + o(1)} \theta^2 x^2 + 48x^2 \frac{o(1)}{1 + o(1)} \theta x + 96x^3 \frac{o(1)}{1 + o(1)} = 0.$$

Since we work only with $x > 0$, we can divide the last equation by x^3 , and we obtain the following equation (with unknown θ):

$$(50) \quad \theta^3 - 6 \frac{\frac{1}{2} + o(1)}{1 + o(1)} \theta^2 + 48 \frac{o(1)}{1 + o(1)} \theta + 96 \frac{o(1)}{1 + o(1)} = 0.$$

Finally, taking into account the equality:

$$\frac{\frac{1}{2} + o(1)}{1 + o(1)} = \frac{1}{2} + o(1)$$

we can write equation (50) in the form:

$$(51) \quad \theta^3 - 3\theta^2 + v_2(x)\theta^2 + v_1(x)\theta + v_o(x) = 0,$$

where $v_1(x)$, $v_2(x)$, $v_o(x)$ are three positive functions defined in a neighbourhood of $+\infty$ and tending to 0 when x tends to $+\infty$. If for a fixed x' we find a number θ' being a root of equation (51), then the number $h' = \theta' \cdot x'$ is a root of equation (41). It is then enough to study equation (51). We shall prove much more. Namely we have the following:

Proposition 25. *For each $\alpha > 0$ there exists a point x_2 such that for each $x > x_2$ equation (53) has in the interval $[-\alpha, \alpha]$ exactly two roots θ_- and θ_+ , and moreover $\theta_- < 0 < \theta_+$.*

Proof. Indeed, Proposition 25 is stronger than Lemma 20, where we need only the existence of a negative root and of a positive root. In Proposition 25 we prove not only that the roots exist, but also that we can find the solutions in an arbitrary open interval containing the origin. Without loss of generality, we may assume, that $\alpha \leq 1$. Let us fix then a positive number $1 \geq \alpha > 0$ and choose x_2 so large, that for $x > x_2$ we have:

$$(52) \quad v_2(x) \cdot \alpha^2 + v_1(x) \cdot \alpha + v_o(x) < 2\alpha^2$$

and

$$(53) \quad v_2(x) \cdot \alpha^2 - v_1(x) \cdot \alpha + v_o(x) < 2\alpha^2,$$

Such an x_2 exists since all three functions v_2 , v_1 , v_o are $o(1)$ when x tends to $+\infty$. Let us fix $x > x_2$. We rewrite equation (51) in the form: $f(\theta) = g(\theta)$, where

$$(54) \quad f(\theta) = \theta^3 + v_2(x) \cdot \theta^2 + v_1(x) \cdot \theta + v_o(x),$$

and

$$(55) \quad g(\theta) = 3 \cdot \theta^2.$$

Let us set $h(\theta) = f(\theta) - g(\theta)$ and let us consider the interval $[0, \alpha]$. We have: $h(0) = f(0) - g(0) = v_o(x) > 0$ and, (since $\alpha < 1$ and using the inequality (52)) we obtain:

$$h(\alpha) = f(\alpha) - g(\alpha) = \alpha^3 + v_2(x) \cdot \alpha^2 + v_1(x) \cdot \alpha + v_o(x) < \alpha^2 + 2\alpha^2 - 3\alpha^2 = 0.$$

Thus equation (51) has a root $\theta_+ \in (0, \alpha)$.

Now we will consider the interval $[-\alpha, 0]$. For $\theta = 0$ we have, as above $h(0) = v_o(x) > 0$. For $\theta = -\alpha$ we have (since $-\alpha^3 < 0$ and we have inequality (53):

$$(56) \quad h(-\alpha) = f(-\alpha) - g(-\alpha) = -\alpha^3 + v_2(x) \cdot \alpha^2 - v_1(x) \cdot \alpha + v_o(x) - 3\alpha^2 <$$

$$(57) \quad < v_2(x) \cdot \alpha^2 - v_1(x) \cdot \alpha + v_o(x) - 3\alpha^2 < 2\alpha^2 - 3\alpha^2 < 0.$$

Once more the continuity argument implies the existence of the root θ_- of the equation (51) in the interval $(-\alpha, 0)$. Let us remark, that $\theta_- \cdot x = h_-^*(x)$ and $\theta_+ \cdot x = h_+^*(x)$. This ends the proof of Proposition 25, hence moreover Lemma 20. □

3.7. The order of magnitude of lenses. By the results of the previous subsection, we can consider four functions: h_- , h_+ , h_-^* and h_+^* , which are defined in an interval (M, ∞) , and such that the following inequalities holds (for each $x \in (M, \infty)$) :

$$(58) \quad h_-^*(x) < h_-(x) < 0 < h_+(x) < h_+^*(x).$$

Our aim is to establish the order of magnitude at $+\infty$ of the difference $H(x) = h_+(x) - h_-(x)$. We will prove the following:

Proposition 26. *The function H satisfies the relation:*

$$H(x) = o(x),$$

when x tends to $+\infty$

Proof. This follows directly from the property formulated in Proposition 25. Indeed, it is sufficient to show separately, that $h_+(x) = o(x)$ and $|h_-(x)| = o(x)$. To prove the first relation, let us fix a positive number $\epsilon > 0$. It follows from Proposition 25 (setting $\alpha = \epsilon$) that there exists $M_1 > M$, such that $x > M_1$ implies, that there exists a number $\theta < \epsilon$ (θ depending on x) such that $h_+^*(x) = \theta \cdot x$. But this means, that

$$\frac{h_+^*(x)}{x} < \epsilon$$

for $x > M_1$. The proof for h_-^* is similar. □

Now we can prove a theorem on the order of magnitude of the length of lenses S_k using the Proposition 26. First we shall prove the following lemma about sequences tending to $+\infty$.

Lemma 27. *Suppose that we have four sequences $(x_k^-)_1^\infty, (x_k^+)_1^\infty, (z_k)_1^\infty$, and $(e_k)_1^\infty$ such that:*

$$(59) \quad 0 < x_k^- \leq e_k < e_{k+1} \leq x_k^+,$$

$$(60) \quad x_k^- \leq z_k \leq x_k^+,$$

$$(61) \quad \lim_{k \rightarrow \infty} e_k = +\infty,$$

$$(62) \quad \lim_{k \rightarrow \infty} \frac{x_k^+ - x_k^-}{z_k} = 0.$$

Then

$$\lim_{k \rightarrow \infty} \frac{e_{k+1} - e_k}{e_k} = 0.$$

Proof. From (60) and (62) we deduce that:

$$\lim_{k \rightarrow \infty} x_k^+ = +\infty.$$

It must be also

$$\lim_{k \rightarrow \infty} x_k^- = +\infty.$$

Indeed, suppose that there exists an infinite subset $\mathbb{L} \subset \mathbb{N}$ and a constant $K > 0$ such that $0 \leq x_n^- \leq K$ for $n \in \mathbb{L}$. Then for $n \in \mathbb{L}$ we have:

$$0 \leq \frac{x_n^+ - K}{z_n} \leq \frac{x_n^+ - x_n^-}{z_n}$$

Hence by (63)

$$\frac{x_n^+ - K}{z_n} \rightarrow 0, n \in \mathbb{L}.$$

This implies that $\lim_{n \in \mathbb{L}} z_n = +\infty$. In consequence

$$\lim_{n \in \mathbb{L}} \frac{x_n^+}{z_n} = 0,$$

thus there exists $n \in \mathbb{L}$ such that $x_n^+ < z_n$, but this is impossible.

From the inequality

$$\frac{x_k^+ - x_k^-}{x_k^+} \leq \frac{x_k^+ - x_k^-}{z_k}$$

we deduce that

$$\lim_{k \rightarrow +\infty} \frac{x_k^-}{x_k^+} = 1$$

and this gives

$$\lim_{k \rightarrow +\infty} \frac{x_k^+ - x_k^-}{x_k^-} = 0.$$

But

$$\frac{x_k^+ - x_k^-}{e_k} \leq \frac{x_k^+ - x_k^-}{x_k^-}$$

then

$$\lim_{k \rightarrow \infty} \frac{x_k^+ - x_k^-}{e_k} = 0.$$

Since

$$\frac{e_{k+1} - e_k}{e_k} \leq \frac{x_k^+ - x_k^-}{e_k}$$

then

$$\lim_{k \rightarrow \infty} \frac{e_{k+1} - e_k}{e_k} = 0,$$

and this ends the proof of Lemma 27. □

Lemma 28. *The graph of the function π^* lies between the graphs of the functions $Li - \varepsilon$ and $Li + \varepsilon$.*

Proof. Suppose the opposite. Then there exist two consecutive prime numbers p_n and p_{n+1} , such that the points $A = (p_n, n)$ and $B = (p_{n+1}, n+1)$ lies between $Li - \varepsilon$ and $Li + \varepsilon$ and the segment $[A; B]$ cuts the graph of $Li - \varepsilon$ or $Li + \varepsilon$. But the subgraph of $Li + \varepsilon$ is convex, then $[A; B]$ cuts only the graph of $Li - \varepsilon$. This means, that there exists a point $x \in (p_n, p_{n+1})$ such that the point $X = (x, n)$ lies below the graph of $Li - \varepsilon$. But $X = (x, \pi(x))$, then from the definition of the error term, X lies between the graphs of $Li - \varepsilon$ and $Li + \varepsilon$. This ends the proof of Lemma 28. \square

Lemma 29. *Let S_k be a lens defined by the extremal prime numbers e_k and e_{k+1} . Then the straight line joining the points $U = (e_k, \pi(e_k))$ and $V = (e_{k+1}, \pi(e_{k+1}))$ cannot cut the graph of $Li - \varepsilon$ in two distinct points.*

Proof. This follows from the Lemma 28 since, by the definition of extremal points, all the graph of π^* lies below the straight line joining the points U and V . \square

The main theorem of this section is the following:

Theorem 30. *With the notations as above if the Riemann Conjecture is true, then*

$$\lim_{k \rightarrow +\infty} \frac{e_{k+1}}{e_k} = 1.$$

Proof. Let U and V be as in Lemma 29. Take the straight line $l(U, V)$ joining U and V and translate it to the position l^* where the straight line l^* is parallel to $l(U, V)$ and tangent to the graph of $Li - \varepsilon$. This line l^* cuts the graph of $Li + \varepsilon$ in points U^* and V^* , whose first coordinates are x_k^- and x_k^+ respectively, and the tangent point is z_k . It is not hard to check, that the sequences $(x_k^-)_1^\infty, (x_k^+)_1^\infty, (z_k)_1^\infty$, and $(e_k)_1^\infty$ satisfy the assumptions of Lemma 27. Then this ends the proof of the theorem. \square

We have an equivalent formulation.

Corollary 31. *The length of lenses $x \rightarrow S(x)$ satisfies the equality $S(x) = o(x)$.*

4. PART III

4.1. Final remarks. It is natural to ask if one can prove the results like Theorem 30 or Corollary 31 without assuming the Riemann Hypothesis. Maybe this is possible, but it seems, that the method used in this paper is insufficient. In particular an analogous argumentation applied to $L(x) = \frac{x}{\ln x}$ and $\varepsilon(x) = C \cdot \frac{x}{\ln^2 x}$ gives only $S(x) = O(x)$. I was also not able to prove Theorem 30 using $L(x) = Li(x)$ and

$$\varepsilon(x) = O \left(x \cdot \exp \left(\frac{A(\ln x)^{\frac{3}{5}}}{(\ln(\ln x))^{\frac{1}{5}}} \right) \right).$$

On the other hand for $L(x) = Li(x)$ the error term $\varepsilon(x) = O(x^\alpha \cdot (\ln^k x))$ ($\alpha > \frac{1}{2}$ and $k \in \mathbb{Z}$) is sufficient.

If one assumes the Riemann hypothesis, then some naive argumentation leads to the equality like $S(x) = O(\sqrt{x} \ln^2 x)$, which seems to be supported by the experimental data. This may suggest, that the problem of determining the right order of magnitude of $S(x)$ at infinity is near to the problem of determining the right order of magnitude of the difference $|Li(x) - \pi(x)|$.

I have no idea about "the small gaps between extremal primes". As it was mentioned in Part I, Question 13, the small gaps between extremal primes -i.e. the small S_k - may occur, but the theorems like for example

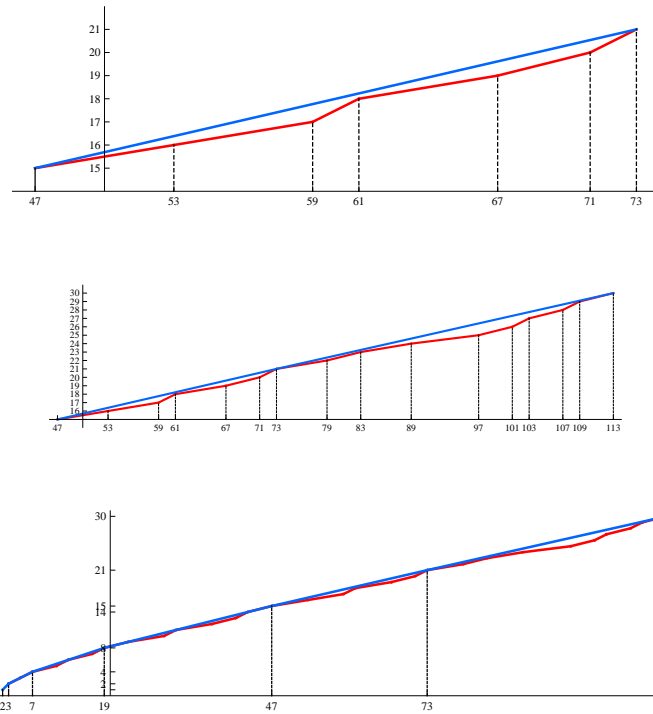
$$\liminf \frac{e_{k+1} - e_k}{\ln e_k} = 0$$

or at least

$$\liminf \frac{e_{k+1} - e_k}{\sqrt{e_k}} = 0$$

seems to be out of reach.

As it was mentioned in Introduction, Montgomery and Wagon in [2] considered the function $M(x) = x \rightarrow \frac{x}{\pi(x)}$. I used an analogous algorithm as in Proposition 4 to obtain about 1500 "another" extremal prime numbers, $(m_k)_1^\infty$ "generated" by the function $M(x)$ instead of $\pi(x)$. Generated by $M(x)$ means, that the points $(m_k, M(m_k))$ are extremal points of the convex hull of the subgraph of the function $M(x)$. Clearly $(m_k)_1^\infty$ and $(e_k)_1^\infty$ are not the same sequences, there are many differences, but on the other hand they behave (in asymptotic sense) similarly.



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